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On gradient Ricci solitons with symmetry

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Joint work with

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Notation

Ricci solitons:
(M, g) pseudo-Riemannian manifold and X a vector field on M satisfying

$$\boxed{\mathcal{L}_X g + Ric = \lambda g}$$

where Ric is the Ricci tensor and $\lambda \in \mathbb{R}$.

- Motivation comes from the study of the *Ricci flow*

$$t \to g(t), \qquad \frac{d}{dt} g(t) = -2Ric(t)$$

(Hamilton 1982)
For any prescribed metric $g(0)$ on a compact manifold M, there exists a unique solution on a maximal interval $[0, T)$, where $0 < T \leq \infty$.

$\longrightarrow$ If the initial metric g_0 is *Einstein* with $Ric_0 = \lambda g_0$, then $g(t) = (1 - 2\lambda t) g_0$

$\longrightarrow$ Ricci solitons are *self-similar solutions* of the Ricci flow

$$g(t) = (1 - 2\lambda t) \varphi_t^* g,$$

where φ_t is the one-parameter group of diffeomorphisms associated to $\frac{1}{1-2\lambda t} X$.

A Ricci soliton is called *expanding*, *steady* or *shrinking* depending on whether $\lambda = \frac{1}{n}(2\mathrm{div} X + Sc)$ satisfies $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$.

Notation

Gradient Ricci solitons:
(M, g) pseudo-Riemannian manifold and $f : M \to \mathbb{R}$ smooth function satisfying

$$\boxed{Ric + Hess_f = \lambda g}$$

where Ric is the Ricci tensor and $\lambda \in \mathbb{R}$.

(Perelman 2002)
- Compact Ricci solitons are gradient.

(Hamilton 1995; Ivey 1993)
- Expanding and steady compact Ricci solitons are trivial (i.e., Einstein)

(Hamilton 1988; Ivey 1993)
- Two and three-dimensional compact Ricci solitons are trivial.

(Koiso 1990; Cao 1991; Wang, Zhu 2004)
- Examples exist in $\mathbb{C}P^2 \sharp k \overline{\mathbb{C}P^2}$, $(k = 1, 2)$.

$\to$ The gradient Ricci soliton equation relates the geometry of (M, g) via its Ricci curvature with that of the level sets of the potential function through its second fundamental form.

$\to$ Nontrivial pseudo-Riemannian gradient Ricci solitons may exist with $\|\nabla f\| = 0$.

Notation

Gaussian soliton:
For any $\lambda \neq 0$, $f(x) = \frac{\lambda}{2}\|x\|^2$ defines a gradient Ricci soliton on $\mathbb{R}^n_\nu$.

The Gaussian soliton is expanding or shrinking depending on the value of λ.

Rigid solitons:
A gradient Ricci soliton is said to be *rigid* if it is isometric to a quotient of $N \times \mathbb{R}^k$, where N is Einstein and $f = \frac{\lambda}{2}\|x\|^2$ on the Euclidean factor.

(Petersen, Wylie; 2009)

- Homogeneous Riemannian gradient Ricci solitons are rigid.
- Complete locally conformally flat Riemannian shrinking GRS are rigid.

(Batat, Brozos-Vázquez, G.-R., Gavino-Fernández; 2011)

- Lorentzian Cahen-Wallach symmetric spaces are non-rigid steady gradient Ricci solitons.

Algebraic solitons: (Lauret)
Expanding solitons on nilpotent and solvable Lie groups.

(Onda; 2011)

- Algebraic Ricci solitons on Lorentzian Lie groups

(Brozos-Vázquez, Calvaruso, G.-R., Gavino-Fernández; 2011)

- Left-invariant Ricci solitons on three-dimensional Lorentzian Lie groups.

  $\rightarrow$ The Ricci operator has a single eigenvalue.

Plan of the talk

1. **Locally homogeneous Lorentzian gradient Ricci solitons**
    - Non-steady locally homogeneous gradient Ricci solitons
    - Steady locally homogeneous gradient Ricci solitons

2. **Riemannian Ricci solitons of constant scalar curvature**
    - Curvature-homogeneous gradient Ricci solitons
    - Four-dimensional Kähler gradient Ricci solitons

1. Locally homogeneous Lorentzian gradient Ricci solitons

- Let (M,g) be a gradient Ricci soliton with constant scalar curvature. If X is a Killing vector field, then $\operatorname{grad} X(f)$ is a parallel vector field.

  Moreover, if $\lambda \neq 0$, then $\operatorname{grad} X(f) = 0$ if and only if $X(f) = 0$.

$$\begin{aligned} 0 = \nabla f(X(f)) &= \nabla_{\nabla f} g(\nabla f, X) = g(\nabla_{\nabla f}\nabla f, X) + g(\nabla f, \nabla_{\nabla f} X) \\ &= \operatorname{Hes}_f(\nabla f, X) + \tfrac{1}{2}(\mathcal{L}_X g)(\nabla f, \nabla f) \\ &= -\operatorname{Ric}(\nabla f, X) + \lambda g(\nabla f, X) = \lambda X(f), \end{aligned}$$

→ If $\operatorname{grad} X(f)$ is timelike/spacelike, then (M,g) splits a one-dimensional factor.

→ If $\operatorname{grad} X(f)$ is null, then (M,g) is a *Walker manifold*.

- Let (M,g) be a locally homogeneous non-steady Lorentzian gradient Ricci soliton. Then it splits as a product $M = N \times \mathbb{R}^k$ for some $k \geq 0$ where either

  (1) (N, g_N) is a Lorentzian Einstein manifold and the soliton is rigid, or

  (2) (N, g_N) is a Lorentzian strictly Walker manifold.

1. Locally homogeneous Lorentzian gradient Ricci solitons

Theorem (Brozos-Vázquez, G.-R., Gavino-Fernández, Gilkey; 2014)
Let (M,g) be a locally homogeneous Lorentzian non-steady gradient Ricci soliton. Then one of the following holds:

(1) (M,g) is irreducible and Einstein.

(2) (M,g) is rigid, this is, there is a local splitting $(M,g)=(N\times\mathbb{R}^k_\nu, g_N+g_e)$ where (N,g_N) is Einstein with Einstein constant λ and $(\mathbb{R}^k_\nu, g_e)$ is pseudo-Euclidean space, $\nu=0,1$.

(3) (M,g) locally splits as

$$(M,g)=(N_0\times N_1\times\mathbb{R}^k, g_0+g_1+g_e)$$

where (N_0,g_0) is an indecomposable locally homogeneous Lorentzian gradient Ricci soliton, (N_1,g_1) is a Riemannian Einstein manifold with Einstein constant λ and $(\mathbb{R}^k,g_e)$ is Euclidean space.

$\longrightarrow$ If $\dim N_0\leq 3$, then the soliton is rigid.

$\longrightarrow$ If $\ker \mathrm{Ric}=2$, then the soliton is rigid.

✓ Locally homogeneous Lorentzian non-steady GRS of dimension ≤ 4 are rigid.

✓ Locally homogeneous Lorentzian non-steady GRS of dimension are rigid if and only if the Weyl tensor is harmonic.

1. Locally homogeneous Lorentzian gradient Ricci solitons

- Some consequences of the gradient Ricci soliton equation

$$\mathrm{Hes}_f + \mathrm{Ric} = \lambda g$$

→ $\boxed{\nabla Sc = 2\,\mathrm{Ric}(\nabla f)}$, and hence, $\mathrm{Ric}(\nabla f, \cdot\,) = 0$ if Sc is constant.

→ In the steady case ($\lambda = 0$), $\mathrm{hes}_f(\nabla f) = 0$, and hence ∇f is a geodesic vector field if Sc is constant.

→ Bochner identity $\frac{1}{2}\Delta g(\nabla f, \nabla f) = \|\,\mathrm{hes}_f\,\|^2 + \mathrm{Ric}(\nabla f, \nabla f) + g(\nabla \Delta f, \nabla f)$ shows that $\lambda(n\lambda - Sc) = \|\,\mathrm{hes}_f\,\|^2$ if Sc is constant.

→ In the steady case ($\lambda = 0$), both hes_f and Ric are isotropic if Sc is constant.

→ $\boxed{Sc + \|\nabla f\|^2 - 2\lambda f = \mathrm{const.}}$, and thus in the steady case ($\lambda = 0$), f is a solution of the *Eikonal equation* $\|\nabla f\|^2 = \mu$, if Sc is constant.

We consider separately the cases $\mu < 0$, $\mu = 0$ and $\mu > 0$.

1. Locally homogeneous Lorentzian gradient Ricci solitons

Theorem (Brozos-Vázquez, G.-R., Gavino-Fernández, Gilkey; 2014)
Let (M, g) be a locally homogeneous steady gradient Ricci soliton such that $\|\nabla f\|^2 = \mu < 0$. Then (M, g) splits locally as a product $\mathbb{R} \times N$, where N is flat and f is the projection on $\mathbb{R}$.

→ Since $\mathrm{hes}_f(\nabla f) = 0$, one may restrict hes_f to $\nabla f^\perp$, which inherits a positive definite metric.

→ Bochner identity

$$\frac{1}{2}\Delta g(\nabla f, \nabla f) = \|\,\mathrm{hes}_f\,\|^2 + \mathrm{Ric}(\nabla f, \nabla f) + g(\nabla \Delta f, \nabla f)$$

and $\Delta f = -Sc$ shows that $\|\,\mathrm{hes}_f\,\|^2 = 0$ and hence $\mathrm{hes}_f = 0$.

→ ∇f is a parallel timelike vector field and thus (M, g) splits locally as $\mathbb{R} \times N$.

→ (N, g_N) is locally homogeneous Ricci flat Riemannian manifold.

(Spiro, 1993)
Riemannian Ricci flat locally homogeneous manifolds are flat.

1. Locally homogeneous Lorentzian gradient Ricci solitons

- Let (M,g) be a homogeneous steady gradient Ricci soliton with $\|\nabla f\|^2 = 0$.

  Then the Ricci operator is two-step or three-step nilpotent.

  Moreover, if the Ricci operator is two-sep nilpotent, then there is a null parallel vector field on (M,g).

  (Calvaruso; 2007)

  A three-dimensional (complete and simply connected) homogeneous space is either symmetric or a Lie group.

- Let (M,g) be a three-dimensional homogeneous steady gradient Ricci soliton with $\|\nabla f\|^2 = 0$. Then (M,g) admits a null parallel vector field.

  $\longrightarrow$ Classification of three-dimensional homogeneous Walker manifolds

Walker manifold: a Lorentzian manifold admitting a parallel null vector field. There exist local coordinates $(x, y, \tilde{x})$ where the metric is given by

$$g(\partial_x, \partial_x) = -2\phi(x,y), \qquad g(\partial_x, \partial_{\tilde{x}}) = g(\partial_y, \partial_y) = 1.$$

(G.-R., Gilkey, Nikčević; 2014)

Let $\mathcal{N}_b$ be defined by $\phi(x,y) = b^{-2}e^{by}$, for $b \neq 0$.

Let $\mathcal{P}_c$ be defined by $\phi(x,y) = \frac{1}{2}y^2\alpha(x)$, where $\alpha_x = c\alpha^{3/2}$, $\alpha > 0$.

Let $\mathcal{CW}_\varepsilon$ be defined by $\phi(x,y) = \varepsilon y^2$.

1. Locally homogeneous Lorentzian gradient Ricci solitons

- Let (M, g) be a three-dimensional homogeneous steady gradient Ricci soliton with $\|\nabla f\|^2 > 0$. Then (M, g) admits a null parallel vector field.

$\rightarrow$ If $\dim(\ker(\mathrm{Ric})) = 1$, then $X = \nabla f$ is left-invariant and $\mathrm{Ric}^2 = 0$.

(Brozos-Vázquez, Calvaruso, G.-R., Gavino-Fernández; 2011)

A three-dimensional Lorentzian Lie group admits a left-invariant Ricci soliton if and only if the Ricci operator has a single eigenvalue.

$\rightarrow$ If $\dim(\ker(\mathrm{Ric})) = 2$, then Ric is two-step nilpotent or diagonalizable.

If Ric is diagonalizable, then $\mathrm{Ric} = \mathrm{diag}[0, 0, Sc]$ and isotropic.
Then $Sc = 0$ and $\mathrm{Ric} = 0$.

(Calviño-Louzao, G.-R., Vázquez-Abal, Vázquez-Lorenzo; 2012)

A three-dimensional Lorentzian homogeneous manifold is Walker if and only if $\mathrm{Ric}^2 = 0$.

$\rightarrow$ Classification of three-dimensional homogeneous Walker manifolds

1. Locally homogeneous Lorentzian gradient Ricci solitons

Theorem (Brozos-Vázquez, G.-R., Gavino-Fernández, Gilkey; 2014)
Let (M, g) be a three-dimensional locally homogeneous Lorentz gradient Ricci soliton. Then one of the following holds

(1) The soliton is trivial

(2) The soliton is rigid

(3) The gradient Ricci soliton is steady and

- (3.1) (M, g) is locally isometric to $\mathcal{CW}_\varepsilon$,
- (3.2) (M, g) is locally isometric to $\mathcal{P}_c$,
- (3.3) (M, g) is locally isometric to $\mathcal{N}_b$.

Walker manifold: a Lorentzian manifold admitting a parallel null vector field. There exist local coordinates $(x, y, \tilde{x})$ where the metric is given by

$$g(\partial_x, \partial_x) = -2\phi(x, y), \qquad g(\partial_x, \partial_{\tilde{x}}) = g(\partial_y, \partial_y) = 1.$$

Let $\mathcal{N}_b$ be defined by $\phi(x, y) = b^{-2}e^{by}$, for $b \neq 0$.

Let $\mathcal{P}_c$ be defined by $\phi(x, y) = \frac{1}{2}y^2\alpha(x)$, where $\alpha_x = c\alpha^{3/2}$, $\alpha > 0$.

Let $\mathcal{CW}_\varepsilon$ be defined by $\phi(x, y) = \varepsilon y^2$.

1. Locally homogeneous Lorentzian gradient Ricci solitons

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(1) The soliton is trivial

(2) The soliton is rigid

(3) The gradient Ricci soliton is steady and

- (3.1) (M, g) is locally isometric to $\mathcal{CW}_\varepsilon$,
- (3.2) (M, g) is locally isometric to $\mathcal{P}_c$,
- (3.3) (M, g) is locally isometric to $\mathcal{N}_b$.

→ GRS are geodesic vector fields and thus complete in $\mathcal{CW}_\varepsilon$ and $\mathcal{N}_b$.

→ GRS in $\mathcal{CW}_\varepsilon$ and $\mathcal{P}_c$ are isotropic, while those in $\mathcal{N}_b$ are spacelike.

→ $\mathcal{CW}_\varepsilon$ and $\mathcal{P}_c$ admit expanding, steady and shrinking RS, while $\mathcal{N}_b$ admits only steady RS.

1. Locally homogeneous Lorentzian gradient Ricci solitons

Let (M,g) be a Lorentzian *locally symmetric* space.

(1) If (M,g) is irreducible, then (M,g) has constant sectional curvature.

(2) If (M,g) is indecomposable but reducible, then (M,g) is a Cahen-Wallach symmetric space.

(N,g_N) is a *Cahen-Wallach symmetric space* if there are coordinates $(t,y,x_1,\ldots,x_n)$ so:

$$g_N = 2\,dtdy + \left(\sum_{i=1}^{n} \kappa_i\, x_i^2\right) dy^2 + \sum_{i=1}^{n} dx_i^2 \qquad \text{for} \qquad 0 \neq \kappa_i \in \mathbb{R}\,.$$

Assume that all $\kappa_i \neq 0$ to ensure that (N,g_N) is indecomposable.

Theorem (Brozos-Vázquez, G.-R., Gavino-Fernández, Gilkey; 2014)
Let (M,g) be a locally symmetric Lorentzian gradient Ricci soliton. Then (M,g) splits locally as a product $M = N \times \mathbb{R}^k$ where

(1) if (M,g) is not steady, then (N,g_N) is Einstein and the soliton is rigid,

(2) if (M,g) is steady, then (N,g_N) is locally isometric to a Cahen-Wallach symmetric space.

2. Riemannian Ricci solitons of constant scalar curvature

A gradient Ricci soliton is said to be *rigid* if it is isometric to a quotient of $N \times \mathbb{R}^k$, where N is Einstein and $f = \frac{\lambda}{2}\|x\|^2$ on the Euclidean factor.

Theorem (Petersen, Wylie; 2009)
Let (M, g) be a complete shrinking or expaning gradient Ricci soliton. If any of the following conditions holds, then the Ricci soliton is rigid,

(1) Sc is constant and the radial curvature $K(\cdot, \nabla f)$ is non-negative or non-positive.

(2) Sc is constant and $0 \leq Ric \leq \lambda g$ or $\lambda g \leq Ric \leq 0$.

(3) $Ric > 0$ or $Ric < 0$ and the radial curvature $K(\cdot, \nabla f)$ vanishes.

$\longrightarrow$ Bounded Ricci curvature implies Einstein in the Lorentzian setting.

$\longrightarrow$ Lorentzian Cahen-Wallach symmetric spaces are non-rigid steady gradient Ricci solitons which have zero scalar curvature and are radially flat.

Theorem (Fernández-López, G.-R.; 2011, Munteanu, Sesum; 2011)
A complete gradient shrinking Ricci soliton is rigid if and only if its Weyl tensor is harmonic.

2. Riemannian Ricci solitons of constant scalar curvature

✓ All known examples of complete GRS with constant scalar curvature are rigid.

Theorem (Petersen, Wylie; 2009)
Let (M, g) be a GRS with constant scalar curvature, then,

(1) If $\lambda = 0$, then $Sc = 0$ and (M, g) is Ricci flat

(2) If $\lambda > 0$, then $Sc \in [0, n\lambda]$.

(3) If $\lambda < 0$, then $Sc \in [n\lambda, 0]$.

Moreover the extreme values are achieved only in the Einstein case.

→ The potential function f is *isoparametric* in the constant scalar curvature case, i.e. $\|\nabla f\|^2 = b(f) = 2\lambda f\,, \qquad \Delta f = a(f) = n\lambda - Sc.$

→ Scalar curvature depends on the dimension of the focal varieties $M_\pm$.

$$Hess_f = 0 \text{ on } TM_\pm \qquad Hess_f = \frac{1}{2} b'(f) \text{ on } TM_\pm^\perp.$$

→ Focal varieties are non-empty submanifolds.

Theorem (Fernández-López, G.-R.; 2014)
Let (M, g) be a non-steady complete gradient Ricci soliton with constant scalar curvature. Then the scalar curvature $Sc = k\lambda$, where $k = 1, \ldots, n-1$.

The value $k = n - 1$ is achieved only in the rigid case.

No complete gradient shrinking soliton exist with scalar curvature $Sc = \lambda$.

2. Riemannian Ricci solitons of constant scalar curvature

Theorem (Petersen, Wylie; 2009)
Any homogeneous gradient Ricci soliton is rigid.

(M, g) is said to be k-*curvature homogeneous* if for each pair of points $p, q \in M$, there is a linear isometry $\Phi_{pq} : T_pM \to T_qM$ such that

$$\Phi_{pq}^* R(q) = R(p), \quad \Phi_{pq} \nabla R(q) = \nabla R(p), \quad \Phi_{pq} \nabla^k R(q) = \nabla^k R(p).$$

Any locally homogeneous manifold is k-curvature homogeneous for all k and the converse holds true if k is sufficiently large.

Theorem (Fernández-López, G.-R.; 2014)
Let (M, g) be a 0-curvature homogeneous complete gradient Ricci soliton. Then it is rigid.

Theorem (Fernández-López, G.-R.; 2014)
Let (M, g) be a complete GRS with constant scalar curvature. Then it is rigid if and only if the Ricci operator has constant rank.

→ Let $R_1, \ldots, R_k$ be nonzero Ricci curvatures. Since the Ricci operator has constant rank, one measures

$$\sum_{i=1}^{k} (R_i - \lambda)^2 = \|Ric\|^2 - 2\lambda Sc + k\lambda^2 = 0 \quad \text{since } \Delta_f Sc = \lambda Sc - \|Ric\|^2.$$

2. Riemannian Ricci solitons of constant scalar curvature

Theorem (Fernández-López, G.-R.; 2014)
Let (M, g) be a non-steady complete gradient Ricci soliton with constant scalar curvature. Then (M, g) is rigid, provided that the Ricci operator has at most three distinct Ricci curvatures.

- ✓ Complete gradient Ricci solitons with constant scalar curvature are rigid in dimension 3.
- ✓ Complete Kähler gradient Ricci solitons with constant scalar curvature are rigid in dimension ≤ 6.
- ✓ Complete gradient Ricci solitons are rigid if and only if the traces of Ric, Ric^2, Ric^3 and Ric^4 are constant.

$$\sum_i R_i^2(\lambda - R_i)^2 = \sum_i R_i^4 - 2\lambda \sum_i R_i^3 + \lambda^2 \sum_i R_i^2$$

Thus $R_i \in \{0, \lambda\}$, which shows that the Ricci curvatures are constant and the soliton is rigid.

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